THEOREM 3 Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one point c in (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

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EXAMPLE 1 Exploring the Mean Value Theorem

Show that the function $f(x) = x^2$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 2]$. Then find a solution c to the equation

t

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

on this interval.

EXAMPLE 2 Exploring the Mean Value Theorem

Explain why each of the following functions fails to satisfy the conditions of the Mean Value Theorem on the interval $[-1, 1]$.

(a) $f(x) = \sqrt{x^2} + 1$

$(x^2)^{1/2} + 1$

$f'(x) = \frac{1}{\cancel{2}}(x^2)^{-1/2} \cdot \cancel{2}x$

$= \frac{x}{\sqrt{x^2}}$

undefined when $x = 0$

$\therefore$ not differentiable on the open interval

(b) $f(x) = \begin{cases} x^3 + 3 & \text{for } x < 1 \\ x^2 + 1 & \text{for } x \geq 1 \end{cases}$

$\lim_{x \to 1^-} x^3 + 3 = 4$

$\lim_{x \to 1^+} x^2 + 1 = 2$

$\therefore$ this function is not continuous at $x = 1$

EXAMPLE 3 Applying the Mean Value Theorem

Let $f(x) = \sqrt{1 - x^2}$, $A = (-1, f(-1))$, and $B = (1, f(1))$. Find a tangent to f in the interval $(-1, 1)$ that is parallel to the secant AB.

$(1-x^2)^{1/2}$

$A = (-1, 0)$ $\quad$ $B = (1, 0)$

slope $\overline{AB}$: $\frac{0-0}{-1-1} = 0$

$f'(x) = \frac{1}{\cancel{2}}(1-x^2)^{-1/2} \cdot -\cancel{2}x$

$= \frac{-x}{\sqrt{1-x^2}}$

$\frac{-x}{\sqrt{1-x^2}} = 0$ $\quad$ $x = 0$

tangent : $y = 1$

Physical Interpretation

If we think of the difference quotient $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that the instantaneous change at some interior point must equal the average change over the entire interval.

EXAMPLE 4 Interpreting the Mean Value Theorem

If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec, or 30 mph. At some point during the acceleration, the theorem says, the speedometer must read exactly 30 mph (Figure 4.15).

DEFINITIONS Increasing Function, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. f **increases** on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
2. f **decreases** on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

$f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ $\quad f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1)$

COROLLARY 1 Increasing and Decreasing Functions

Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.
2. If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

EXAMPLE 5 Determining Where Graphs Rise or Fall

The function $y = x^2$ (Figure 4.16) is

$y' = 2x$ never undefined

$2x = 0$

$y' = 0$ when $x = 0$ $\quad$ $-\ |\ +$ (0)

y is increasing $(0, \infty)$ because $y' > 0$

y is decreasing $(-\infty, 0)$ because $y' < 0$

EXAMPLE 6 Determining Where Graphs Rise or Fall

Where is the function $f(x) = x^3 - 4x$ increasing and where is it decreasing?

$f'(x) = 3x^2 - 4$ never undefined

$3x^2 - 4 = 0$

$3x^2 = 4$

$x = \pm\sqrt{\frac{4}{3}}$

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$-\sqrt{\frac{4}{3}}$ $+\sqrt{\frac{4}{3}}$

f(x) is increasing $\left(-\infty, -\sqrt{\frac{4}{3}}\right)$ $\left(\sqrt{\frac{4}{3}}, \infty\right)$ because $f'(x) > 0$

f(x) is decreasing $\left(-\sqrt{\frac{4}{3}}, \sqrt{\frac{4}{3}}\right)$ because $f'(x) < 0$

COROLLARY 2 **Functions with $f' = 0$ are Constant**

If $f'(x) = 0$ at each point of an interval I, then there is a constant C for which $f(x) = C$ for all x in I.

COROLLARY 3 **Functions with the Same Derivative Differ by a Constant**

If $f'(x) = g'(x)$ at each point of an interval I, then there is a constant C such that $f(x) = g(x) + C$ for all x in I.

EXAMPLE 7 **Applying Corollary 3**

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

$$f(x) = -\cos x + C$$

$$2 = -\cos 0 + C$$

$$2 = -(1) + C$$

$$C = 3$$

$$\boxed{f(x) = -\cos x + 3}$$

EXAMPLE 7 **Applying Corollary 3**

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

DEFINITION **Antiderivative**

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The process of finding an antiderivative is **antidifferentiation.**

EXAMPLE 8 **Finding Velocity and Position**

Find the velocity and position functions of a body falling freely from a height of 0 meters under each of the following sets of conditions:

(a) The acceleration is 9.8 m/sec^2 and the body falls from rest.

$a = 9.8$

(b) The acceleration is 9.8 m/sec^2 and the body is propelled downward with an initial velocity of 1 m/sec.

a) $v(t) = 9.8t + C$ $\quad v(t) = 9.8t$

$0 = 9.8(0) + C$

$C = 0$

$s(t) = 4.9t^2 + C$ $\quad s(t) = 4.9t^2$

$0 = 4.9(0) + C$

$C = 0$

b) $v(t) = 9.8t + C$ $\quad v(t) = 9.8t + 1$

$1 = 9.8(0) + C$

$C = 1$

$s(t) = 4.9t^2 + t + C$ $\quad s(t) = 4.9t^2 + t$

$0 = 4.9(0) + 0 + C$

$C = 0$